PP36723. Proposed by Rovsen Pirkuliev.

If $x > 0$, prove that $e^x + e^{-x} + \pi^x + \pi^{-x} > \dfrac{1}{\arctan e^x} + \dfrac{1}{\arctan \pi^{-x}}$

Solution by Arkady Alt, San Jose, California, USA.

We will prove that for any $t \in \left(0, \dfrac{\pi}{2}\right)$ holds inequality $\tan t + \cot t > \dfrac{1}{t}$.

Since $\tan t + \cot t = \dfrac{2}{\sin 2t}$ then $\tan t + \cot t > \dfrac{1}{t} \Leftrightarrow \dfrac{2}{\sin 2t} > \dfrac{1}{t} \Leftrightarrow 2t > \sin 2t$

where latter inequality holds because $x > \sin x$ for any $x \in (0, \pi)$.

Then by replacing t in inequality $\tan t + \cot t > \dfrac{1}{t}$ with $\arctan e^x$ and $\arctan \pi^{-x}$

we obtain, respectively, $\tan(\arctan e^x) + \cot(\arctan e^x) > \dfrac{1}{\arctan e^x} \Leftrightarrow$

$e^x + e^{-x} > \dfrac{1}{\arctan e^x}$ and $\tan(\arctan \pi^x) + \cot(\arctan \pi^x) > \dfrac{1}{\arctan \pi^{-x}} \Leftrightarrow$

$\pi^{-x} + \pi^x > \dfrac{1}{\arctan \pi^{-x}}$.

Hence, $e^x + e^{-x} + \pi^x + \pi^{-x} > \dfrac{1}{\arctan e^x} + \dfrac{1}{\arctan \pi^{-x}}$.

Remark.

For any $a > 0$ holds inequality $a^x + a^{-x} > \dfrac{1}{\arctan a^x}$.